Languages and compatible relations on monoids *

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Abstract

With every monoid, one can associate various binary relations, in particular the so-called principal quasi-order, principal order and principal congruence. The aim of this paper is to show that any compatible quasi-order, order and congruence can be obtained as an intersection of the corresponding principal relations. The particular case of languages -subsets of free monoids generated by an alphabet- is also considered.

1 Introduction

Let M be a monoid with identity 1. Several types of binary relations can be associated with a given subset of M. In particular, the left quotient, right quotient and quotient induce the notions of principal quasi-order, order and equivalence.

Right principal and principal equivalences were first studied in the theory of semigroups respectively in [3] and [2]. They have been more or less rediscovered and used later in the theories of automata and formal languages where they play a significant role. Right principal congruences have been used first in [9] to characterize general right congruences as intersections of right principal congruences. These relations frequently appear also in the theory of automata and formal languages where the monoid considered is the free monoid X^* generated by an alphabet X. Namely, they are the tools used for obtaining algebraic characterizations of several classes of languages (see the Myhill-Nerode characterization of regular languages).

The purpose of this paper is to show that, by using similar techniques as in [9], it is possible to give a unified description and charaterize all the above compatible relations as intersections of the corresponding principal relations.

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The paper is organized as follows. In Section 2, the different notions of quotient associated with a subset of a monoid are introduced and used for the definition and the study of principal quasi-orders, orders and congruences. In Section 3 we consider the principal quasi-order, order and congruence associated with a language over a given alphabet, and we recall some well known results. The last section is devoted to showing how to characterize compatible quasi-orders, orders and congruences as intersections of the corresponding principal relations.

As general references, see for example [1] for the theory of semigroups and [7], [8] for the theory of automata and formal languages.

2 Principal quasi-orders, orders and congruences on monoids

Let M be a monoid with identity 1. A binary relation ρ on M is called:

– a *quasi-order* or a *pre-order* if ρ is reflexive and transitive;

– an *equivalence* if ρ is a symmetric quasi-order;

– an *order if* ρ is an anti-symmetric quasi-order.

The binary relation ρ is called *trivial* if it is the identity, that is, if $u\rho v$ implies $u = v$. It is said to be *compatible (right compatible, left compatible)* if $u\rho v$ implies

$$xuy\ \rho\ xvy\ (uy\ \rho\ vy,\ xu\ \rho\ xv)\ \forall x, y \in M.$$

An equivalence relation ρ that is compatible (right compatible, left compatible) is said to be a *congruence (right congruence, left congruence)*.

Let ρ be a congruence of M and let M/ρ be the set of classes of ρ. If $[u]$ denotes the class modulo ρ containing u, then the product $[u][v]$ of two classes $[u]$ and $[v]$ is the class $[uv]$ containing the element uv. This product is associative and the set M/ρ is a *monoid* called the *quotient monoid* of M modulo ρ. The mapping $\phi : M \to M/\rho$ defined by $\phi(u) = [u]$ is a morphism of M onto M/ρ.

With every subset $L \subseteq M$, one can associate three different kinds of quotients:

(i) the *right quotient* $u^{-1}L$ of L by $u \in M$:

$$u^{-1}L = \{x \in M | ux \in L\};$$

(ii) the *left quotient* Lu^{-1} defined symmetrically by:

$$Lu^{-1} = \{x \in M | xu \in L\};$$

(iii) the *quotient* $L..u$ of L by u:

$$L..u = \{(x, y) | x, y \in M, xuy \in L\}.$$

Proposition 2.1 *Let L be a subset of M. Then:*

(i) $u^{-1}L \subseteq v^{-1}L \Rightarrow (ux)^{-1}L \subseteq (vx)^{-1}L$ *for all* $x \in M$;

(ii) $Lu^{-1} \subseteq Lv^{-1} \Rightarrow L(xu)^{-1} \subseteq L(xv)^{-1}$ *for all* $x \in M$;

(iii) $L..u \subseteq L..v \Rightarrow L..xuy \subseteq L..xvy$ *for all* $x, y \in M$.

Proof. *(i)* If $r \in (ux)^{-1}L$, then $uxr \in L$ and $xr \in u^{-1}L \subseteq v^{-1}L$. Hence $vxr \in L$, $r \in (vx)^{-1}L$ and $(ux)^{-1}L \subseteq (vx)^{-1}L$.

(ii) Symmetric to the proof of *(i)*.

(iii) If $(r, s) \in L..xuy$, then $rxuys \in L$, $(rx, ys) \in L..u \subseteq L..v$, $rxvys \in L$ and therefore $(r, s) \in L..xvy$. Hence $L..xuy \subseteq L..xvy$. □

Corollary 2.1 *Let $x \in M$. Then $u^{-1}L = v^{-1}L$ implies $(ux)^{-1}L = (vx)^{-1}L$, $Lu^{-1} = Lv^{-1}$ implies $L(xu)^{-1} = L(xv)^{-1}$ and $L..u = L..v$ implies $L..xuy = L..xvy$ for all $x, y \in M$.*

The left quotient, right quotient and quotient are used in defining the principal quasi-order, order and congruence associated with every subset $L \subseteq M$, as follows:
(i) the relations σ_L and ${}_L\sigma$ are defined by

$$u\sigma_L v \;\Leftrightarrow\; u^{-1}L \subseteq v^{-1}L, \quad u\,{}_L\sigma v \;\Leftrightarrow\; Lu^{-1} \subseteq Lv^{-1},$$

(ii) the relations ρ_L and ${}_L\rho$ by

$$u\rho_L v \;\Leftrightarrow\; u^{-1}L = v^{-1}L, \quad u\,{}_L\rho v \;\Leftrightarrow\; Lu^{-1} = Lv^{-1},$$

(iii) the relation P_L and S_L by

$$uP_L v \;\Leftrightarrow\; L..u = L..v, \quad uS_L v \;\Leftrightarrow\; L..u \subseteq L..v.$$

Proposition 2.2 *Let L be a subset of M. Then:*

(i) The relations σ_L and ${}_L\sigma$ are respectively right and left compatible quasi-orders on M.

(ii) The relations ρ_L and ${}_L\rho$ are respectively right and left congruences on M.

(iii) The relation P_L is a congruence on M and the relation S_L is a compatible quasi-order on M.

Proof. *(i)* It is immediate that σ_L and ${}_L\sigma$ are reflexive and transitive relations and therefore quasi-orders. Let $u\sigma_L v$ and $x \in M$. Then $u^{-1}L \subseteq v^{-1}L$ and, by Proposition 2.1 *(i)*, $(ux)^{-1}L \subseteq (vx)^{-1}L$. Hence $ux\sigma_L vx$. The proof for ${}_L\sigma$ is similar.

(ii) The relations ρ_L and ${}_L\rho$ are clearly reflexive, symmetric and transitive, hence equivalence relations. If $u \equiv v\ (\rho_L)$ and $x \in M$, then $u^{-1}L = v^{-1}L$ and, by Corollary 1.2, $(ux)^{-1}L = (vx)^{-1}L$. Hence $ux \equiv vx\ (\rho_L)$ and ρ_L is a right congruence. The proof is similar for ${}_L\rho$.

(iii) The relation P_L is clearly an equivalence relation. If $u \equiv v\ (P_L)$ and if $x, y \in M$, then, by Corollary 2.1, $L..xuy = L..xvy$. Therefore $xuy \equiv xvy\ (P_L)$ and P_L is a congruence.

It is immediate that S_L is a quasi-order. By Proposition 2.1, uS_Lv implies $L..xuy \subseteq L..xvy$, that is, $xuyS_Lxvy$. Hence S_L is compatible. □

The equivalence ρ_L $({}_L\rho)$ is called the *right (left) principal congruence* and the congruence P_L is called the *principal congruence* defined by the subset $L \subseteq M$.

The quasi-orders $\sigma_L({}_L\sigma)$ and S_L will be called respectively the *right (left) principal quasi-order* and the *principal quasi-order* defined by the subset L of M.

A subset $L \subseteq M$ of the monoid M is called *(right) disjunctive* if (ρ_L) P_L is the equality, i.e. $(u^{-1}L = v^{-1}L)$ $L..u = L..v$ implies $u = v$. For example if M is a group, then every element of M is (right) disjunctive.

The following result shows that the notion of disjunctive subset naturally arises when considering quotient monoids of the form M/P_L, where L is any subset of M.

Proposition 2.3 *Let L be a nonempty subset of the monoid M and let $\bar{M} = M/P_L$ be the quotient monoid modulo the principal congruence P_L. If $\bar{L} = \{[u] | u \in L\}$ where $[u]$ is the class of u modulo P_L, then $\bar{L}$ is a disjunctive subset of $\bar{M}$.*

Proof. Suppose that $[u] \equiv [v]\ (P_{\bar{L}})$. This means that $[x][u][y] = [xuy] \in \bar{L}$ if and only if $[x][v][y] = [xvy] \in \bar{L}$. Since L is a union of classes of P_L, $xuy \in L$ if and only if $xvy \in L$. Hence $u \equiv v\ (P_L)$ and $[u] = [v]$, that is $\bar{L}$ is disjunctive in $\bar{M}$. □

If σ is a compatible quasi-order on the monoid M, then it is well known that the relation $\bar{\sigma}$ defined by $u\bar{\sigma}v$ iff $u\sigma v$ and $v\sigma u$ is a congruence on M. Furthermore the relation τ induced on $\bar{M} = M/\bar{\sigma}$ by $\bar{u}\tau\bar{v}$ iff $u\sigma v$ is a compatible order on the monoid $\bar{M}$.

Proposition 2.4 *If L is a disjunctive subset of the monoid M, the quasi-order S_L is a compatible order of M.*

Proof. If uS_Lv and vS_Lu, then $L..u \subseteq L..v$ and vice-versa. Hence $L..u = L..v$ and $u = v$, which shows that S_L is antisymmetric. □

Remark that this compatible order can be trivial. For example if M is a group and $u \in G$, then u is disjunctive. However the compatible quasi-order $S_{\{u\}}$ is the equality.

3 The particular case of languages

Let X be an alphabet (finite or infinite) and let X^* be the free monoid generated by X. Elements and subsets of X^* are called respectively *words* and *languages* over X. The length of a word $u \in X^*$ is denoted by $|u|$. Since X^* is a monoid, all the results of Section 2 can be applied to X^*.

If $L \subseteq X^*$ is a language over X, the principal right congruence and the principal congruence are called respectively the *right syntactic* and the *syntactic congruences* of the language L. The quotient monoid X^*/P_L is called the *syntactic monoid* of L and denoted by $syn(L)$.

The converse problem, of determining when a given monoid M is isomorphic to the syntactic monoid of some language, is connected to the existence of a disjunctive subset in M. We recall the following result and give a detailed proof.

Proposition 3.1 *If $L \subseteq X^*$ is a language over X, then the syntactic monoid syn(L) contains a disjunctive subset. Conversely, if a monoid M contains a disjunctive subset D, then there exist a language L over some alphabet X such that M is isomorphic to syn(L).*

Proof. The first part follows from Proposition 2.3. For the converse, let X be a set of generators of M (for example M itself) and let X^* be the free monoid generated by X.

Let $\phi : X^* \to M$ be the canonical mapping of X^* onto M defined by $\phi(x_1x_2\cdots x_k) = x_1x_2\cdots x_k$. This is a morphism of X^* onto M. The equivalence relation θ defined on X^* by $u \equiv v\ (\theta)$ iff $\phi(u) = \phi(v)$ is a congruence of X^* such that X^*/θ is isomorphic to M.

Let $L = \phi^{-1}(D)$. We will show in the following that $\theta = P_L$. If $u \equiv v\ (\theta)$ and if $(x, y) \in L..u$, then $xuy \in L$ and $\phi(xuy) \in D$. From

$$\phi(xuy) = \phi(x)\phi(u)\phi(y) = \phi(x)\phi(v)\phi(y) = \phi(xvy)$$

follows $\phi(xvy) \in D$, hence $xvy \in L$. Therefore $L..u \subseteq L..v$. By symmetry, we have $L..v \subseteq L..u$. Hence $L..u = L..v$, $u \equiv v\ (P_L)$, that is, $\theta \subseteq P_L$.

Let us show that $P_L \subseteq \theta$. Suppose that $u \equiv v\,(P_L)$, $u, v \in X^*$ with u not equivalent to v modulo θ. Then $\phi(u) \neq \phi(v)$. Since D is disjunctive, P_D is the equality on M, hence $D..\phi(u) \neq D..\phi(v)$. Consequently, there exist $x, y \in M$ such that $x\phi(u)y \in D$ and $x\phi(v)y \notin D$ (or vice versa). Let $r, s \in X^*$ such that $\phi(r) = x$, $\phi(s) = y$. Then $\phi(rus) \in D$ and $\phi(rvs) \notin D$, i.e. $rus \in L$, $rvs \notin L$, a contradiction because $L..u = L..v$. Therefore $P_L \subseteq \theta$ and we have the equality $\theta = P_L$.

Since $X^*/P_L = X^*/\theta$ is isomorphic to M, we conclude that M is isomorphic to the syntactic monoid syn(L) of the language L. □

When the alphabet X is finite, the syntactic congruence and the syntactic monoid can be used to give algebraic characterizations of some classes of languages.

Recall that a language L over the finite alphabet is *regular* if it is accepted by a finite automaton. The following result, often called the Myhill-Nerode theorem, gives an algebraic characterization of regular languages.

Proposition 3.2 *(see for example [4]) Let L be a language over a finite alphabet X. Then the following properties are equivalent:*

– *L is regular;*

– *the number of classes of the (right) syntactic congruence (ρ_L) P_L is finite;*

– *the syntactic monoid of L is finite.* □

We give now an example of a language L that is not regular and describe the classes of its syntactic congruence.

Let ρ be the equivalence of X^* having as its classes the languages $X^n, n \geq 0$. This equivalence is a congruence because it is compatible. It is possible to find a language L such that $\rho = \rho_L$, i.e. ρ is a right syntactic congruence. Let $L = \{w \in X^* | \ |w| = 2^n, n \geq 1\}$. The language L is reflective ($uv \in L$ implies $vu \in L$), hence ${}_L\rho = \rho_L = P_L$. Clearly $\rho \subseteq \rho_L$. Suppose that $\rho \neq \rho_L$. Then there exist u, v with $|u| < |v|$ such that $u \equiv v\ (\rho_L)$. Let $k = |v|$ and $i = |v| - |u|$. Then there exists $x \in X^*$ such that $ux \in X^{2^k} \subseteq L$. Consequently, $vx \in L$. On the other hand, $vx \in X^{2^k+i}$, which implies that $2^k + i = 2^{k+r}$and $i = 2^{k+r} - 2^k = 2^k(2^r - 1)$. Since $r > 0$, then $i \geq 2^k$, $i + |u| = |v|$ and $k = |v| > 2^k$, a contradiction. Therefore $\rho = \rho_L$ and, since $\rho_L = P_L$, ρ is the syntactic congruence of L.

Since P_L has infinitely many classes, L is not regular. Remark that the syntactic monoid $syn(L)$ of L is isomorphic to the monoid $(\mathbf{N}, +)$.

If a language L is not regular, it is still possible that some classes of its syntactic congruence P_L are regular. Such a situation is exemplified below.

The *residue* $W(L)$ of a language L is the language

$$W(L) = \{u \in X^* | L..u = \emptyset\}.$$

In other words, the residue contains those words which are not subwords of any word of the language. Clearly, if not empty, $W(L)$ is a class of P_L and, if L is regular then $W(L)$ is also regular.

The language $L = \{a^n b^n | n \geq 0\}$ over $X = \{a, b\}$ is context-free, but not regular. However its residue $W(L) = X^* \backslash a^* b^*$ is regular.

The following proposition shows that there is a large class of languages having a regular residue. Recall that a language L is *commutative* iff $u \in L$ implies that L contains all words obtained by arbitrarily permuting the letters of u. A language is comutative iff its syntactic monoid is commutative.

Proposition 3.3 *If L is a commutative language, then its residue $W(L)$ is regular.*

Proof. If $W(L) = \emptyset$, this is trivial. Suppose $W(L) \neq \emptyset$ and let $uv \in W(L)$. If $vu \notin W(L)$, then there exist $x, y \in X^*$ such that $xvuy \in L$. Since L is commutative, this implies $xuvy \in L$. We arrived at a contradiction, as uv was a word in $W(L)$. Therefore $uv \in W(L)$ implies $vu \in W(L)$. Let $uv \in W(L)$ and $x \in X^*$. Then $vu \in W(L)$, $vux \in W(L)$ and $uxv \in W(L)$. Therefore $W(L)$ is a μ-ideal. (Recall that a μ-ideal is a subset $I \subseteq X^*$ with the property: $u = u_1u_2 \in I$ implies $u_1xu_2 \in I$ for all $x \in X^*$.) Since every μ-ideal is regular ([10]), $W(L)$ is regular. □

In the remaining part of this section we give some examples of quasi-orders and orders associated with languages.

The language $L = \{a^nb^n | n \geq 1\}$ is context-free but not regular. The relation σ_L is a right compatible quasi-order, but not an order, because for example, $b^{-1}L = (b^2)^{-1}L = \emptyset$, hence $b\sigma_L b^2$ and $b^2\sigma_L b$.

Let $X = \{a, b\}$ and define the following order $\leq$ on X^*:

– if $|u| < |v|$, then $u \leq v$

– if $|u| = |v|$, then $\leq$ is the lexicographic order.

Let $X^* = \{x_1, x_2, \cdots, x_n, \cdots\}$ be the listing of the words of X^* with respect to the order $\leq$. Define the language L by:

$$L = \{u_1 = x_1, u_2 = x_1x_2, \cdots, u_n = x_1x_2\cdots x_n, \cdots\}$$

It is easy to see that P_L is the identity and therefore L is disjunctive. However ρ_L is not the identity, because the set $W_L = \{u \in X^* | u^{-1}L = \emptyset\}$ is not empty and W_L is an infinite class of ρ_L. The relation σ_L is a nontrivial right compatible quasi-order. However σ_L is not an order relation. For example, for $u \in W_L$, $v \in L$ we have $u\sigma_L v$, but not $v\sigma_L u$, which means σ_L is not symmetric. If $u, v \in W_L$, $u \neq v$, then $u\sigma_L v$ and $v\sigma_L u$ because $u^{-1}L = v^{-1}L = \emptyset$.

If a language L is disjunctive, then the relation S_L is a compatible order relation that can be trivial. For example, let Q be the set of all primitive words of X^*, $\mathrm{card}(X) \geq 2$. (Recall that a word $w \in X^+$ is called primitive if $w = g^i$, $g \in X^+$, implies $i = 1$.) Then Q is disjunctive, i.e. $Q..u = Q..v$ implies $u = v$. It follows then that the principal or syntactic congruence P_Q is the identity. Furthermore the relation $Q..u \subseteq Q..v$ implies $u = v$ and hence the quasi-order S_Q is also the identity.

A disjunctive language L such that there exist $u, v \in X^*$ with $L..u \subseteq L..v$ and $u \neq v$ is called a *m-disjunctive language* (see [6]). It is immediate that in such a case the relation S_L is a compatible order that is not trivial. In the following, we show how some m-disjunctive languages can be constructed (see [6]).

Let $X = \{a_1, a_2, \cdots, a_r\}$ and let f be the mapping of X^* into N defined by:

$$f(1) = 0,\ f(a_i) = i\ (1 \leq i \leq r)$$

$$f(a_{i_1}a_{i_2}\cdots a_{i_k}) = f(a_{i_1})(r+1)^{k-1} + f(a_{i_2})(r+1)^{k-2} + \cdots + f(a_{i_{k-1}})(r+1) + f(a_{i_k})$$

Let

$$L_i = \{u\bar{a}_i^k | u = u'a_i, u' \in X^*, f(u) \leq k\},\ 1 \leq i \leq r\},$$

where $\bar{a}_i = a_{i+1}$ for $1 \leq i \leq r-1$ and $\bar{a}_r = a_1$.

As shown in [6], the language L_i is m-disjunctive for any $1 \leq i \leq r$, hence S_L is a nontrivial compatible order.

4 Characterization of compatible relations

Let σ be a quasi-order on M. An *upper section* (or *starting section)* of σ is a nonempty subset $S \subseteq M$ such that $u \in S$ and $u\sigma x$ implies $x \in S$. For every $u \in M$, the set $[u) = \{x \in M | u\sigma x\}$ is an upper section of σ called the *monogenic upper section* generated by u.

One can easily define, by symmetry, the notions of *lower section* and *monogenic lower section.* Note that all the following results are valid when replacing upper section with lower section.

Lemma 4.1 *If σ is a compatible (right compatible) quasi-order of M and if $[u)$ is the monogenic upper section generated by u, then $\sigma \subseteq S_{[u)}$ ($\sigma \subseteq \sigma_{[u)}$).*

Proof. Suppose that $r\sigma s$ and let $(x, y) \in [u)..r$. Then $xry \in [u)$, that is, $u\sigma xry$. Since σ is compatible, $xry\sigma xsy$ which implies $u\sigma xsy$. Consequently, $xsy \in [u)$, $(x, y) \in [u)..s$, i.e. $[u)..r \subseteq [u)..s$. Therefore $rS_{[u)}s$ and $\sigma \subseteq S_{[u)}$. The proof for the case of right compatible orders is similar. □

Let $\Lambda = \{L_i | i \in I\}$ be a family of subsets $L_i \subseteq M$. The relations S_Λ and σ_Λ are defined on M by:

$$uS_\Lambda v \ \Leftrightarrow\ L_i..u \subseteq L_i..v \quad \text{for all } i \in I, \quad u\sigma_\Lambda v \ \Leftrightarrow\ u^{-1}L_i \subseteq v^{-1}L_i \quad \text{for all } i \in I.$$

Clearly, $S_\Lambda = \bigcap_{i\in I} S_{L_i}$ and $\sigma_\Lambda = \bigcap_{i \in I} \sigma_{L_i}$.

Proposition 4.1 *Let $\Lambda = \{L_i | i \in I\}$ be a family of subsets of M. Then S_Λ (σ_Λ) is a compatible (right compatible) quasi-order on M. Conversely, if σ is a compatible (right compatible) quasi-order on M, then there exists a family $\Lambda = \{L_i | i \in I\}$ of subsets of M such that $\sigma = S_\Lambda$ ($\sigma = \sigma_\Lambda$).*

Proof. We will consider only the case of compatible quasi-order, the other one being similar. Since the quasi-orders S_{L_i} are compatible, their intersection S_Λ is also a compatible quasi-order.

For the converse, let $\Lambda = \{L_i | i \in I\}$ the set of all the monogenic upper sections L_i of σ. By Lemma 4.1, $\sigma \subseteq \bigcap_{i \in I} S_{L_i} = \tau$. Suppose that $\sigma \neq \tau$. Then there exist $r, s \in M$ such that $r\tau s$ and $r \not\sigma s$. If $K = [r)$, then $r \in K$ and $1 \in r^{-1}K$. Since $K \in \Lambda$, we have $rS_K s$, which implies $K..r \subseteq K..s$. Consequently, $1 \in s^{-1}K$, $s \in K$ and $r\sigma s$, a contradiction. We conclude that $\sigma = \tau$ and $\sigma = S_\Lambda$. □

Let X^* be the free monoid generated by the alphabet X. The following relation λ:

$$u \ \lambda \ v \ \Leftrightarrow \ |u| \leq |v|$$

is a compatible quasi-order on X^*. The upper section $[u)$ of λ generated by u is given by $[u) = \{v \in X^* | \ |u| \leq |v|\}$. If $|u| = n$, let $L_n = [u)$. Then it is easy to see that $\lambda = \bigcap_{n \geq 1} S_{L_i}$.

If we aim the relation S_Λ (σ_Λ) to be a compatible (right compatible) order, we have to impose the condition that the family of subsets $\{L_i | i \in I\}$ is disjunctive.

A family $\Lambda = \{L_i | i \in I\}$ of subsets $L_i \subseteq M$ is said to be *disjunctive (right disjunctive)* if S_Λ *(*σ_Λ*)* is the identity relation, i.e. if $L_i..u = L_i..v$ ($u^{-1}L_i = v^{-1}L_i$) for all $i \in I$ implies $u = v$. If the family contains a unique subset $L \subseteq M$, then L is called a *disjunctive (right disjunctive) subset.* For example, the family consisting of the unique subset Q, the set of all the primitive words of M, is disjunctive and right disjunctive.

Proposition 4.2 *Let* $\Lambda = \{L_i | i \in I\}$ *be a disjunctive (right disjunctive) family of subsets of* M*. Then* S_Λ *(*σ_Λ*) is a compatible (right compatible) order on* M*. Conversely, if* σ *is a compatible (right compatible) order on* M*, there exists a disjunctive (right disjunctive) family* $\Lambda = \{L_i | i \in I\}$ *of subsets of* M *such that* $\sigma = S_\Lambda$ *(*$\sigma = \sigma_\Lambda$*).*

Proof. We will prove only the case of compatible order, the other one being similar. Since each S_{L_i} is a compatible quasi-order, the intersection S_Λ of these compatible quasi-orders is also a compatible quasi-order. If $uS_\Lambda v$ and $vS_\Lambda u$, then $L_i..u = L_i..v$, hence $u = v$ because of the disjunctivity property. Therefore S_Λ is a compatible order.

For showing the converse, let $\Lambda = \{L_i | i \in I\}$ be the set of all the monogenic upper sections L_i of σ. Since σ is a compatible order, hence a compatible quasi-order, by Proposition 4.1 and its proof, $\sigma = \bigcap_{i \in I} S_{L_i} = S_\Lambda$. The family Λ is disjunctive, because $L_i..u \subseteq L_i..v$ and $L_i..v \subseteq L_i..u$ for all $i \in I$ implies $u\sigma v$ and $v\sigma u$. Since σ is an order, this implies $u = v$. □

As expected, as the relation P_L is a congruence relation, results similar to Propositions 4.1, 4.2 hold also for congruences, respectively right congruences.

Let $\Lambda = \{L_i | i \in I\}$ be a family of subsets $L_i \subseteq M$. The relations ρ_Λ and P_L are defined on M by:

$$u\rho_\Lambda v \ \Leftrightarrow \ u^{-1}L_i = v^{-1}L_i \ \ \forall i \in I, \ \ uP_\Lambda v \ \Leftrightarrow \ L_i..u = L_i..v \ \ \forall i \in I,$$

that is, $\rho_\Lambda = \bigcap_{i \in I} \rho_{L_i}$ and $P_\Lambda = \bigcap_{i \in I} P_{L_i}$.

Proposition 4.3 *Let* $\Lambda = \{L_i | i \in I\}$ *be a family of subsets of* M*. Then* P_Λ *(*ρ_Λ*) is a congruence (right congruence) on* M*. Conversely, if* ρ *is a congruence (right congruence) on* M*, then there exists a family* $\Lambda = \{L_i | i \in I\}$ *of subsets of* M *such that* $\rho = P_\Lambda$ *(*$\rho = \rho_\Lambda$*).*

Proof. Let us prove, for example, the case of congruences. Since each P_{L_i} is a congruence, the intersection P_Λ of these congruences is also a congruence.

For the converse, let $\Lambda = \{L_i | i \in I\}$ be the set of all the classes L_i of the congruence ρ and let $\tau = \bigcap_{i \in I} P_{L_i}$. Since $\rho \subseteq P_{L_i}$ for all $i \in I$, we have that $\rho \subseteq \tau$. Suppose that $\rho \neq \tau$. Then there exist $u, v \in X^*$ such that $u \equiv v(\tau)$ and $u \not\equiv v(\rho)$. If $[u]$ is the class of u modulo ρ, then $u \in [u]$ and $(1,1) \in [u]..u$. Since $[u] \in \Lambda$, $u \equiv v(P_{[u]})$ and $[u]..u = [u]..v$. Hence $(1,1) \in [u]..v$, $1.v.1 = v \in [u]$, which implies $u \equiv v(\rho)$ – a contradiction. Therefore $\rho = \tau$. □

We conclude by an example showing that, in some cases, a congruence of a monoid can be viewed either as a principal congruence or as the intersection of infinitely many principal congruences. Let X^* be the free monoid over the alphabet X and let ρ be the congruence of X^* having as its classes the languages $X^n, n \geq 0$. In Section 3, we have shown that $\rho = \rho_L = P_L$ is the principal congruence defined by the language $L = \{w \in X^* | \, |w| = 2^n, , n \geq 1\}$.

Let $L_n = X^n$. Then clearly $\rho_{L_n} = P_{L_n}$, because $uv \in L_n$ implies $vu \in L_n$. If $\Lambda = \{L_i | i \geq 0\}$, then it is easy to see that $\rho = \bigcap_{i \geq 0} \rho_{L_i} = \bigcap_{i \geq 0} P_{L_i}$.

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